Empiricism as Unifying Theme in the Standards for Mathematical Practice

Glenn Stevens
Department of Mathematics
Boston University

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On the correlation of mathematics with science

John Dewey's Laboratory School at the University of Chicago introduced an experiential philosophy of education based on "hands-on" experiences at the *elementary* level.

But, in 1899, Dewey wrote:

"The first person who succeeds in working out the real correlation of mathematics with science and advanced form of manual training, will have done more to simplify the problems of *secondary* education than any other one thing that I can think of."

On the correlation of mathematics with science

E. H. Moore's "Response"

In 1903, a few short years after Dewey's "challenge," E. H. Moore, who is often referred to as the father of American mathematics, introduced his own famous laboratory method into the teaching of mathematics—at the *secondary* level. His method included graphical calculation and experimentation in geometry.

While this experiment was eventually abandoned, the underlying ideas that E. H. Moore introduced continue to be influential in American mathematics via the work of his students, R. L. Moore and A. E. Ross, among others.

Induction versus Deduction

(1) E.H. Moore's Laboratory Method

"Why should not the student be directed each for himself to set forth a body of geometric fundamental principles on which he would proceed to erect his geometric edifice? This method would be thoroughly practical and at the same time thoroughly scientific. The various students would have different systems of axioms, and the discussions thus arising in the minds of all: precisely what are the functions of the axioms in the theory of geometry."

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(2) Charles Dodgson's theorem: Every triangle is isosceles!

AUTHENTIC MATHEMATICAL EXPERIENCE

Experience first:

"It has been observed in every human activity experience comes first, and as this experience grows the need for communication motivates the development of language. Sadly enough, in our classroom practice we place language first and experience second. We worry about what we should say in order to help the student 'understand.' By this we mean to provide the effect of experience through the use of suitably chosen words. Not unexpectedly, the effect is at best a very pale image of the real thing."

Arnold Ross

Beliefs about the nature of mathematics

- Mathematics is natural
  - The empirical nature of mathematics
  - People do mathematics naturally
- Mathematics exists independent of us
  - We can perform experiments
  - We can test ideas and decide for ourselves
- Experience precedes formality
  - “Meaning” is determined by experience
  - Definitions and theorems are capstones
  - Language is a tool for coming to terms with experience
- Mathematics is the science of structure
  - Operations, order
  - Shape
  - Continuity
  - Transformation
- Mathematics is the art of figuring things out

THE FOCUS ON MATHEMATICS ACADEMY

The Focus on Mathematics Academy is a Collaboration of

- Boston University, Boston College, UMass Lowell
- Education Development Center, Inc. (EDC)
- Math for America Boston (MfAB)
- Boston area school districts

working to support a community of

- Teachers,
- Educators, and
- Mathematicians

to close the gap between school mathematics and mathematics as a scientific discipline.

The Focus on Mathematics Academy

A community of mathematical practice that has been evolving for almost 25 years.

- A Targeted Math and Science Partnership (MSP)
  - established in 2003
  - with funding by the National Science Foundation
- Rooted in *PROMYS for Teachers*, initiated in 1989 and deeply influenced by Arnold Ross.
- Phase II research beginning in 2009
- DRK12 funding to develop measures of secondary teachers' algebraic habits of mind
- Noyce Projects at Boston University and Boston College with MfA Boston in support of Master Teacher Leaders

WORK TO DATE: *Focus on Mathematics* PROGRAMS

The Focus on Mathematics Academy has established

- **Study Groups**
- **Seminars and colloquia**
- A graduate degree program at Boston University
  - Master of Mathematics for Teacing (MMT)
- Summer Institutes – e.g. **PROMYS**
- Mathematics fairs and Mathematics Expo for students
- Research collaboratives
- Tools for Assessing Secondary Teachers Algebraic Habits of Mind
- **Teachers in leadership roles throughout**

TYPICAL FEATURES OF AN FoM MATHEMATICAL EXPERIENCE

- Engagement in mathematics
- Teachers working together with each other, with educators, and with mathematicians
- The central role of experience
    - empirical basis of mathematical knowledge
    - personal experience as guide for exploration
- Sharing ideas with others
    - in writing
    - in seminars
- Questioning answers
- Low threshold – high ceiling

Some Odd Sums

Some Odd Sums

Some Odd Sums

$$1 = 1$$

Some Odd Sums

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ \ldots & \quad \ldots \end{aligned}$$

Some Odd Sums

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \end{aligned}$$

Some Odd Sums

$$
\begin{aligned}
1 &= 1 \\
1+3 &= 4 \\
1+3+5 &= 9 \\
1+3+5+7 &= 16
\end{aligned}
$$

Some Odd Sums

$$
\begin{aligned}
1 &= 1 \\
1+3 &= 4 \\
1+3+5 &= 9 \\
1+3+5+7 &= 16 \\
1+3+5+7+9 &= 25
\end{aligned}
$$

Some Odd Sums

$$
\begin{aligned}
1 &= 1\\
1+3 &= 4\\
1+3+5 &= 9\\
1+3+5+7 &= 16\\
1+3+5+7+9 &= 25\\
1+3+5+7+9+11 &= 36
\end{aligned}
$$

Some Odd Sums

$$
\begin{aligned}
1 &= 1 \\
1+3 &= 4 \\
1+3+5 &= 9 \\
1+3+5+7 &= 16 \\
1+3+5+7+9 &= 25 \\
1+3+5+7+9+11 &= 36 \\
1+3+5+7+9+11+13 &= 49 \\
\cdots & \quad \cdots
\end{aligned}
$$

Some Odd Sums

$$
\begin{aligned}
1 &= 1 \\
1+3 &= 4 \\
1+3+5 &= 9 \\
1+3+5+7 &= 16 \\
1+3+5+7+9 &= 25 \\
1+3+5+7+9+11 &= 36 \\
1+3+5+7+9+11+13 &= 49 \\
\ldots & \quad \ldots \\
1+3+5+7+9+11+13+\cdots+(2n-1) &= ??
\end{aligned}
$$

Some Odd Sums

$$
\begin{aligned}
1 &= 1 \\
1+3 &= 4 \\
1+3+5 &= 9 \\
1+3+5+7 &= 16 \\
1+3+5+7+9 &= 25 \\
1+3+5+7+9+11 &= 36 \\
1+3+5+7+9+11+13 &= 49 \\
\ldots &\quad \ldots \\
1+3+5+7+9+11+13+\cdots+(2n-1) &= n^2
\end{aligned}
$$

A Sample Seminar

(About 15 teachers attended a recent seminar at BU)

The morning session: Teachers talking about their classroom experiences.

- Catching up
  - Informal conversation over morning coffee and bagels.
  - Comparing notes on what's going on in different districts/schools
- Teachers gave presentations about activities they have recently used in their classrooms

The afternoon session: Work together on common topic: in this case, recursively defined functions on the non-negative integers.

THE MORNING SESSION:

A Lawrence Middle School Teacher talked about some problems she showed her students about sums of consecutive integers:

In how many ways can a positive integer be written as a sum of consecutive positive integers?

Experimentation: If $r(n) =$ the number of ways ..., then

$$\begin{aligned}
r(2) &= 1: & 2 &= 2\\
r(3) &= 2: & 3 &= 3 = 1+2\\
r(4) &= 1: & 4 &= 4\\
r(5) &= 2: & 5 &= 5 = 2+3\\
r(6) &= 2: & 6 &= 6 = 1+2+3\\
r(9) &= 3: & 9 &= 9 = 2+3+4 = 4+5\\
r(15) &= 4: & 15 &= 15 = 4+5+6\\
& & &= 1+2+3+4+5 = 7+8\\
r(45) &= 6:
\end{aligned}$$

The Morning Session:

A Boston Teacher wrote the following on the board:

$$\sqrt{(2i)^3} \cdot \sqrt{2i} = \sqrt{(2i)^3} \cdot \sqrt{(2i)^1} = \sqrt{(2i)^4} = \sqrt{16} = 4$$

and also

$$\sqrt{(2i)^3} \cdot \sqrt{2i} = (2i)^{3/2} \cdot (2i)^{1/2} = (2i)^{3/2+1/2} = (2i)^2 = -4$$

What's going on? How can $\sqrt{(2i)^3} \cdot \sqrt{2i}$ be both 4 and -4?
How can I make sense of this for my students?

THE MORNING SESSION:

A Los Angeles Teacher talked about her experience doing topology at PCMI the previous summer

- Posed questions about what one gets by gluing the edges of a square following various rules;
- Showed us “Dirac’s belt trick”.

THE AFTERNONON SESSION:

Exploration of recursively defined functions

- Recurrence
  - Arithmetic and Geometric sequences
  - Use technology to generate data
- Linear Recurrence
  - Fibonacci-like sequences

$$\begin{aligned} f(0) &= 2 \\ f(1) &= 3 \\ f(n) &= 3f(n-1) - 2f(n-2) \quad \text{if } n \geq 2 \end{aligned}$$

The sequence looks like this:

$$2,\ 3,\ 5,\ 9,\ 17,\ 33,\ 65,\ 129,\ \ldots,$$

Sums of Cubes

Sums of Cubes

Sums of Cubes

$$1^3 = 1$$

Sums of Cubes

$$
\begin{aligned}
1^3 &= 1 \\
1^3 + 2^3 &= 9
\end{aligned}
$$

Sums of Cubes

$$\begin{aligned} 1^3 &= 1 \\ 1^3 + 2^3 &= 9 \\ 1^3 + 2^3 + 3^3 &= 36 \end{aligned}$$

Sums of Cubes

$$
\begin{aligned}
1^3 &= 1 \\
1^3 + 2^3 &= 9 \\
1^3 + 2^3 + 3^3 &= 36 \\
1^3 + 2^3 + 3^3 + 4^3 &= 100
\end{aligned}
$$

Sums of Cubes

$$\begin{aligned} 1^3 &= 1 \\ 1^3 + 2^3 &= 9 \\ 1^3 + 2^3 + 3^3 &= 36 \\ 1^3 + 2^3 + 3^3 + 4^3 &= 100 \\ 1^3 + 2^3 + 3^3 + 4^3 + 5^3 &= 225 \end{aligned}$$

Sums of Cubes

$$
\begin{aligned}
1^3 &= 1 \\
1^3 + 2^3 &= 9 \\
1^3 + 2^3 + 3^3 &= 36 \\
1^3 + 2^3 + 3^3 + 4^3 &= 100 \\
1^3 + 2^3 + 3^3 + 4^3 + 5^3 &= 225 \\
1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 &= ??
\end{aligned}
$$

Sums of Cubes

$$
\begin{aligned}
1^3 &= 1 \\
1^3 + 2^3 &= 9 \\
1^3 + 2^3 + 3^3 &= 36 \\
1^3 + 2^3 + 3^3 + 4^3 &= 100 \\
1^3 + 2^3 + 3^3 + 4^3 + 5^3 &= 225 \\
1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 &= 441 \\
\ldots & \quad \ldots \\
1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + \cdots + n^3 &= ??
\end{aligned}
$$

Sums of Cubes

$$
\begin{aligned}
1^3 &= 1^2 \\
1^3 + 2^3 &= (1+2)^2 \\
1^3 + 2^3 + 3^3 &= (1+2+3)^2 \\
1^3 + 2^3 + 3^3 + 4^3 &= (1+2+3+4)^2 \\
1^3 + 2^3 + 3^3 + 4^3 + 5^3 &= (1+2+3+4+5)^2 \\
1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 &= (1+2+3+4+5+6)^2 \\
\ldots &\quad \ldots \\
1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + \cdots + n^3 &= (1+2+3+\cdots+n)^2
\end{aligned}
$$

THE HALL OF 20,000 CEILING LIGHTS – CAR TALK

RAY: Imagine, if you will, that you have a long, long corridor that stretches out as far as the eye can see. In that corridor, attached to the ceiling are lights that are operated with a pull cord. There are gazillions of them, as far as the eye can see. Let's say there are 20,000 lights in a row. They're all off. Somebody comes along and pulls on each of the chains, turning on each one of the lights. Another person comes right behind, and pulls the chain on every second light.
TOM: Thereby turning off lights 2, 4, 6, 8 and so on.
RAY: Right. Now, a third person comes along and pulls the cord on every third light. That is, lights number 3, 6, 9, 12, 15, etcetera. Another person comes along and pulls the cord on lights number 4, 8, 12, 16 and so on. Of course, each person is turning on some lights and turning other lights off. If there are 20,000 lights, at some point someone is going to come skipping along and pull every 20,000th chain.

When that happens, some lights will be on, and some will be off. Can you predict which ones will be on? Think you know?

Decimal Expansions of Rational Numbers

Every rational number has a decimal expansion that either terminates or is eventually periodic. Find the decimal expansion of each of the following rational numbers:

- $\frac{1}{3}, \frac{2}{3};$
- $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7};$
- $\frac{1}{13}, \frac{2}{13}, \frac{3}{13}, \ldots, \frac{12}{13};$
- $\frac{1}{89}, \frac{1}{9899}, \frac{1}{998999}, \ldots$

Pascal's Triangle

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

(1) What is the sum of the entries in the 7th row (i.e. the row at the bottom)? the 6th row? the 5th row? Any conjectures?

(2) Repeat the above experiment but this time take the alternating sum of each row. What do you see now?

(3) How many of the entries in the 7th row are odd? How many in the 6th row? the 5th row? Can you guess how many entries of the 100th row of Pascal's triangle are odd? How many entries in the 100th row do you think will be prime to 3? to 5? What's going on?

A Problem from George Polya

ABRACADABRA: In how many ways can one spell out "ABRACADABRA" by traversing the following diamond, always going from one letter to an adjacent one?

```
          A
        B   B
      R   R   R
    A   A   A   A
  C   C   C   C   C
A   A   A   A   A   A
  D   D   D   D   D
    A   A   A   A
      B   B   B
        R   R
          A
```

Thank You!

THANK YOU!

Visit our websites:
www.focusonmath.org
www.promys.org

Visit us in Boston this summer:
June 28 to August 8, 2015