GETTING INFORMATION

The word “research” has different meanings. School research projects typically involve a student learning about a topic by reading literature in the library and on-line and then writing a report that summarizes what has been learned—it is a search for something already known. The Making Mathematics project emphasizes the other meaning of “research”—the creation and exploration of original problems and results. This latter type of research can benefit from the former variety. Students often pose questions that are variants of better known problems. In such cases, their work is enhanced if they find out what mathematicians already know about these related questions, what techniques were useful in solving the problems, and what open questions remain. In this way, students can build their research on established ground, apply proven approaches, make connections, and prove theorems using earlier work. These efforts let students experience how mathematics grows as a field.

Because mathematics research is typically an “open-book” experience, students need to learn how to find information in textbooks, in the library, and on-line. Researching needed information includes a range of tasks that require persistence and creativity. Although the Internet has become an amazing repository of information, there is still a great deal of information that can be found only in journals and books. Beyond that, there is information and advice that can be gleaned only from fellow mathematicians. Student skills include being able to interview people; search through library stacks; track down leads in bibliographies; accumulate potential keywords for subject indices, Internet searches, and library card catalogues (real or virtual); and read through technical literature.

If a question is original for a student, but not for the field, we may want to encourage that student to avoid the library and just work on her own. Unless she is planning to enter a science fair that requires an original question, a student should avoid a literature review that would reveal a solution to her question. In this case, after she has solved her main problem, she can pose new extensions that move beyond the initial setting and then study what others have done. This after-the-fact search lets the student discover the broader context of her work.

Getting students access to good mathematics resources is a challenge. The Internet is the most available resource and a search via Google or the Math Forum will turn up information on almost any mathematics topic that interests a student. The Resources chapter lists a number of extensive web sites that students can use.

Whenever possible, students should also consult print materials such as journals and books. Most school and local public libraries do not have good mathematics selections. Students who can get to a major public library or a college library will fare better. Some colleges will allow students to use a library without borrowing privileges, so researchers should bring a few dollars in change in order to copy articles that they can then read in a more methodical fashion at home. Primary source searches and journals are very difficult for secondary school students. Secondary sources such as journals by mathematics organizations are often challenging but accessible because they are written for a broader audience.

In addition to wanting to learn about work that is related to our own area of investigation, we also read mathematics articles for several other reasons:

- To learn new mathematics.
- To learn about how others have approached a problem. To be exposed to creative and rigorous mathematical presentations.
- To get new problem ideas.
- To see if our problem has already been solved.
- To find out about results upon which we can build our own proofs.
- To become sympathetic to the challenges that readers of our own writing will face!

READING TECHNICAL LITERATURE

Once students find relevant material from web sites, articles, and books, they need to read it. Even mathematics articles that are accessible are rarely a simple read. Students need to learn that reading a primary source article about original mathematics is not like reading a work of fiction. Although an article may convey a story line (the process by which the author made their discovery), it is more likely a purified reordering of the mathematics. In either case, a strictly linear reading of the definitions, examples (if you are lucky), conjectures, and proofs is not likely to be effective in providing a full understanding of the ideas.

Class Activity: Explain to the class that during the year and in the future they will be reading mathematics written by each other and by others within the mathematical community. Explain that reading technical material within mathematics requires an approach that differs from how they might read other types of literature. This activity works well with an article that is just on the edge of what the students know and can make sense of. The material should not be too easy, but it must also build on familiar content. It should include a mixture of English, symbolic statements, and diagrams so that students practice looking for relationships between the three. The article Triangles with Integer Sides and Sharing Barrels works particularly well with students who have had an Algebra II course or its equivalent (Special thanks to David Singmaster and the MAA for permission to distribute this article). For younger secondary students, articles or units from the periodicals and books listed in the Resources chapter may meet these criteria.

Have the students take turns reading one sentence at a time out loud. After each sentence ask if everyone is sure that they understand what was said. When a sentence does elicit confusion, ask the students for ideas on how they might gain a better understanding of the line. Let their suggestions be the impetus for fleshing out and attempting the various recommendations below. Ask what they would need to do in order to fully understand a given definition or claim that arises. Point out that some students are drawing pictures or generating test cases in order to understand an idea.

Do not continue until everyone is sure that she understands each line. If you feel that people are not voluntarily confessing their doubts or are not aware of their incomplete grasp of a claim, ask them to generate examples that match the information or to explain a statement in their own words.

Typically, a first reading might take from 30 to 60 minutes per page of text. It is worth pointing out to a class that has grappled successfully and valiantly at such a pace how much patience difficult material requires, how much they have learned during that time, and how different reading mathematics papers is from non-technical reading. Even a relatively short article may take too many days to read entirely in class. If the class has managed to identify a number of helpful reading strategies and grappled with a significant section of a paper, it will not diminish the experience if they do not read the rest as a group. Students can complete an article for homework (reasonably at a page or so per night) and show their written questions, examples, comments, and summary as evidence of their work.

The following list provides advice, some of which students should be able to develop from their class reading experience and the rest of which should be shared with them, that students can practice implementing.

Things to do as you read:

- Have a pen and paper handy.
- Keep a running list of the symbols, words, variables, and functions and their meanings so you do not have to flip back and forth looking for information as much. A glossary of definitions and a legend for symbols can increase the fluidity of your reading and make sure that you highlight important ideas.
- Read each sentence and make sure that you understand it. If you do not, figure out why. Is there a word that you do not understand? Does a claim not make sense? Can you draw a picture, carry out a calculation, or create an example that will help you get a better grasp of the idea? Can you refer to an earlier part of the text for clarification?
- Try specific cases. (For the Singmaster article, ask students to generate all solutions to the barrel problem for $N = 7$ and $N = 8$ and to list all integral triangles with perimeters 1 through 10. Are they considering degenerate triangles such as those with side lengths of 1, 3, and 4?).
- When you encounter an unfamiliar word, try to grasp its meaning from context and from the word's prefixes and suffixes. Use a dictionary to look up the word or word parts. A mathematics dictionary, such as The Harper Collins Dictionary of Mathematics by E. J. Borowski and J. M. Borwein, is quite helpful.
- If theorems and conjectures use vocabulary that is unfamiliar, try substituting the words' definitions in their place. This switch will produce a longer but perhaps easier-to-follow statement. Simply replacing a word by what it means often helps you make progress on a problem. For example, "a quadrilateral's centroid is at the intersection of its bimedians" becomes "a quadrilateral's center of mass is at the intersection of the segments connecting opposite midpoints of its sides."
- If an article lacks an abstract, you can try a brisk read of it in order to get an overview of the ideas, figure out where the authors are heading, and then do a more methodical second pass to fill in missed details and connections. Do not become intimidated if you do not understand

what you have read quickly and assume that you will not understand the ideas when you follow their development more completely.

- Often, mathematics articles do not present all of the details of a proof or derivation. Sometimes the missing steps are left "as an exercise for the reader" or they are hinted at with a "which leads to" or a "which can be shown." These gaps are not always so straightforward a task to fill, but they are an unavoidable reality given the limitations of space in a journal or an author's desire to avoid having lengthy computations distract us from their main message. Students should try to write in the steps or missing justifications ("How did they get from A to B? Oh, they subtracted and factored.").

  For example, on the first page of Triangles with Integer Sides and Sharing Barrels, three equations are presented and then we read "A little observation and manipulation shows that [the system] implies $e_i = f_i$ …". Before they proceed, students should verify that they can derive the result themselves. This problem, which follows from elimination of h_i or N in the equations, may stump students unused to solving systems with subscripts because they do not think of "h_i" as a single variable.
- Take notes on important results and methods that might aid your work and new questions that occur to you as you read.

Here are some additional questions to answer as you read:

- What is the goal or main question of the investigation?
- How are examples used and how are they organized?
- What previous work inspired this effort? What background information provided results or methods that the author(s) built upon in their work?
- What variables did they identify and how did they vary or eliminate them?
- Can I outline the line of reasoning used to reach the main conclusions? If so, do it!

For a detailed discussion of questions to ask when reading a new definition or theorem, see Understanding a Definition in the Definitions section and Evaluating and Testing Conjectures in Examples, Patterns, and Conjectures.

One way to encourage students to read challenging articles is to have them readily available. A magazine rack with issues from stimulating journals or a bookcase with Martin Gardner collections and a range of well-written textbooks on both familiar and unfamiliar subjects can lead to pleasure reading in mathematics. Student extracurricular reading can contribute greatly to students' appreciation for the variety of mathematics topics and expand their understanding of problem-solving and proof methods.

There are times when a student's research takes her to the edge of a subject (e.g., group theory or linear algebra) that she may never have heard of before. In such cases, we can tell our student that we see this possible connection. Armed with the reading skills noted above, she can borrow a text and work her way through the lessons that may further her thinking about her question.

INTERNET AND JOURNAL RESEARCH

Because the Internet is an organic collection of individual's interests, there is still a great deal of unevenness in the mathematics writing that is available. Nonetheless, there is such an incredible volume of information that is available that the Internet has now become the single best resource for a secondary student. Searches can proceed via search engines that find matching words and phrases (e.g., Google) or via smaller, but more focussed, catalogs of links that make it possible to browse by subject (e.g., the Math Forum's Internet Mathematics Library).

Electronic resources and search engines generally do not reference writings that are more than a few years old. To find information and articles that date from before the 1990s, a student may need to consult print resources. Students with topics that are not too advanced should begin by looking at several years of the Readers' Guide to Periodical Literature. This guide indexes articles from popular newspapers and magazines (e.g., Scientific American). A student who wanted readable articles on fractals, for example, would have some success with this research tool.

The main resources for finding technical mathematics papers are the Mathematical Reviews (published by the AMS) and Mathematics Abstracts (Zentralblatt Math). Both are organized according to the Mathematics Subject Classification. Unlike the Readers Guide and the indices used by some other disciplines, these references do not allow one to look up articles according to keywords appearing in the title. Researchers must know within which areas of mathematics their work falls. These guides are appropriate for sophisticated high school students with substantial teacher or librarian help. Students should read an abstract to help determine whether to locate the actual article. Abstracts should not be used as the source of information itself.

For searches that do involve keywords, students should expect to take an iterative approach to developing a good list of words related to their investigation. They may begin with two or three words that describe their topic, but then find, as they read their first articles, other words that appear regularly. They should then go back and check the indices and search engines using these words as well. When working with search engines, students should distinguish between searches for collections of words and searches for specific phrases. For example, *Isosceles Right Triangle* (a search for three word) and *"Isosceles Right Triangle"* (a search for that exact phrase) will not produce the same list of links.

Sometimes it can be a challenge to identify the right words or expressions that mathematicians use to describe an idea. Here are some approaches that students can use when refining their searches:

- Try brainstorming as many terms with as similar or related a meaning as possible.
- Try variations on words (e.g., "locus" and "loci" or "proof" and "prove").
- Think of subset and superset ideas. For example, if you are interested in triangles, look up equilateral triangles or polygons.

Once students are in the library stacks, encourage them to browse. Books and journals that are near those that turned up in a search often prove useful even thought they did not turn up in the search themselves. Remind students to copy all relevant bibliographic information from their sources—it is time-consuming having to go back to gather such information after the fact. Not only is bibliographic information a requirement so that readers can track down a source for themselves, it is also how students give credit for ideas that influenced their work. Students should be encouraged to consult the bibliographies of the papers that they read as possible additional resources. Just as they click from Internet site to site via links, researchers can use bibliographies to help find information that direct searches missed.

CONDUCTING AN INTERVIEW

Written resources are not a student's only option and, sometimes, are not an option at all—not all knowledge is written down. When a student's project involves an original question, the best resource may be a mathematician who can reflect on the particulars of the student's problem, offer advice, and help her find useful resources. Consulting with a mathematician may produce first-hand information and insights that are not available in the literature. This contact may also lead to the mathematician becoming an interested audience later on for the student's work, but students should begin by planning a single interview from which they can get the most information. Possible interview candidates include a knowledgeable pre-college teacher, a member of the mathematics department at a local college, or a mathematician who works in industry (e.g., a nearby technology firm, financial institution, or manufacturing facility).

We can help students prepare for an interview before they contact a potential interviewee. Start by having your class read an interview and then analyze the questions, their purpose, and their effectiveness. Sample interviews can be found in *Mathematical people: profiles and interviews* (Donald Albers and G. Alexanderson (1985). Boston, MA: Birkhauser). Non-mathematical interviews may serve equally as well for the task (see, for example, the Globe Magazine interview with Christian Moneyhun at http://www.boston.com/globe/magazine/4-15/interview.shtml or the Mother Jones magazine interview with Margaret Atwood at http://www.motherjones.com/mother_jones/JA97/visions.html).

Once the class has identified some of the features of a good question or sequence of questions, divide them up into groups of five students. Each group should pick one classmate whom they consider to be an expert on some topic (one of our students was an expert on shoes). While the expert goes on-line to hone their knowledge, the remaining four students take 20 minutes to write a series of interesting, probing questions that they will use to interview that expert. Each group then takes a turn interviewing their expert in front of the rest of the class who offers critiques. Common issues that arise include whether interviewers seem to be listening well (and not interrupting), asking follow-up questions that respond to what has been said, whether questions are designed to elicit explanations or just one word answers, and whether the interviewer is calm, serious, and focused in her demeanor.

Once students have some general understandings about what types of interactions and approaches work well in interviews, you can focus their thinking on the specifics of their interviewing tasks. The outline below can help them with this process.

- Contact your interviewee-to-be by phone (or email) and introduce yourself, explain what you are doing, why you would like an interview, and ask if they would be willing to set up a time for you to interview them in person, by phone (although this makes it more difficult to show any work that you have undertaken), or with back-and-forth emails (which creates a record of your exchanges).
- Before the interview, prepare several questions. Consider open-ended questions ("If you were at this point in this problem, what different directions would you consider exploring?") as well as factual questions ("Do you know if this problem has been solved by someone else already?"). You may only have 30 minutes or so for your interview, so select a small number of questions that address your particular needs and be sure to begin with those. Make sure that these are not questions that you could easily answer with your own library or Internet research. In order to make the most of the experience, expect to take the lead during the interview (do not leave it to the interviewee to set the agenda).
- At the start of the interview, make sure that you give a concise introduction to the questions that you are exploring and the progress that you have made thus far. Note any important assumptions that you have made and major obstacles that you have encountered. Ask if there are avenues that you should pursue that you might not be aware of and if there are any mathematics topics that you should study that would be helpful.
- During the interview, listen carefully and ask questions that are connected to what your interviewee has said. Take notes on essential ideas and information that you want to remember.
- At the end of the interview, thank your interview subject for meeting with you. After the interview, send a thank you note.
- Record the date and place of the interview and full name of the person being interviewed for your bibliographic citations.

