DELVING deeper

Allen Olsen

Divisibility Tests

Al Cuoco recently invited Paul Goldenberg to talk to an ongoing study group for high school teachers at Lawrence (Massachusetts) High School, of which I was a member.

Our study group had spent most of fall 2004 looking at patterns in the repeating decimal representations of rational numbers. Our topic on that day was divisibility tests, which we had found useful in determining the repeat length of these representations. Goldenberg and I gave presentations and compared notes about two different but complementary ways of looking at divisibility tests. He recently wrote an article for this department ("How Does One Know If a Number Is Divisible by 17?" *Mathematics Teacher* 99 [April 2006]: 502–5) about this encounter, describing a set of divisibility tests he had developed at an early age. This article is a companion to Goldenberg's.

I first started thinking about divisibility tests in late 1986 in a completely unrelated context. I was, of course, aware from childhood of the standard divisibility tests (e.g., if the sum of the digits of a number is divisible by 3, then so is the number; if a number ends in 5 or 0, then it is divisible by 5). But these rules had an ad hoc quality to them—there was this rule for 3, a different one for 5, and a different one for 11. On this particular day, in a sudden flash of insight, I realized that there was a general algorithm (or "meta-method") for constructing divisibility tests for *nearly all integers.* Not only that, but as I looked further into the algorithm I found out that these divisibility tests were not unique. In other words, for each integer, the number of such tests was potentially unlimited.

For years that is where my insight stood. I used these tests for speeding up calculations or as parlor tricks, but for a long time they eluded application. Then, when I returned to teaching mathematics, I found myself dusting off these tests and using them in applications like factoring and prime number searches. It was thus natural that they should come up in our study group discussion of the repeat lengths of the decimal representations of rational numbers.

An example of the kinds of repeating decimal representations we had been looking at is 1/41, which has a decimal equivalent of $0.\overline{02439}$. In this case the "repeat length" of the decimal representation is 5. To see how this is related to the divisibility properties of 41, write

$$\begin{aligned}\frac{1}{41} &= 0.\overline{02439}\\ &= \frac{2439}{100000}\left(1+\frac{1}{100000}+\left(\frac{1}{100000}\right)^2+\cdots\right)\\ &= \frac{2439}{100000}\left(\frac{1}{1-\frac{1}{100000}}\right)\\ &= \frac{2439}{99999}\\ &= \frac{271}{11111}\end{aligned}$$

The fact that the five-digit number 11111 is divisible by 41 is thus intimately related to fact that its decimal representation has a repeat length of 5. Our study group was therefore led to ask whether we could determine the divisibility of 11111 by 41 without actually doing the division.

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The answer is yes, as Goldenberg showed in his article. Divisibility tests can be applied recursively. You start with 11111. Each step generates a new test number. The new test number is smaller than the previous one. Eventually, the test number will fall within the range of the memorized multiplication tables. When divisibility of the result is determined, then so is the divisibility of the original number.

One test for divisibility by 41 involves taking the units digit, multiplying it by 4, and subtracting it from the rest of the number. This is shown below.

1111|1	Multiply the units digit (1) by 4 and subtract from 1111
– 4	
110|7	Multiply the units digit (7) by 4 and subtract from 110
– 28	
8|2	Multiply the units digit (2) by 4 and subtract from 8
– 8	
0	

Since the result, 0, is divisible by 41, so is each number going up the chain; that is, 82, 1107, and 11111 are all divisible by 41.

Our study group discussed a whole series of divisibility tests, some of which are summarized below. The rest of this article discusses how to generate divisibility tests for many integers. Let's start with tests for divisibility by primes.

Table 1 lists all the odd primes p (except 5) less than 100. To test a number n for divisibility by a prime p, multiply the units digit of n by any of the factors k and add (+) that result to or subtract (–) it from the rest of the number, the part of n that remains after the units digit is removed.

Table 1

Primes Less than 100 and Factors Needed for Divisibility Tests

p	k
3	+ 1, + 4, + 7, – 2, – 5, – 8
7	– 2, + 5
11	– 1, + 10
13	– 9, + 4
17	– 5, + 12
19	+ 2, – 17
23	+ 7, – 16
29	+ 3, – 26
31	– 3, + 28
37	– 11, + 26
41	– 4, + 37
43	+ 13, – 30
47	– 14
53	+ 16
59	+ 6
61	– 6
67	– 20
71	– 7
73	+ 22
79	+ 8
83	+ 25
89	+ 9
97	– 29

"TWO OUT OF THREE" THEOREM

Suppose now that m is any integer, not necessarily prime. The proofs of divisibility tests we consider below all depend on a result that our study group called the "Two out of Three" theorem.

TWO OUT OF THREE THEOREM. *Let three integers A, B, and C obey the equation $A = B + C$ and let the integer m divide any two of $\{A, B, C\}$. Then the integer m will divide the third as well.*

PROOF. Without affecting the proof's generality we can take B and C as the two integers divisible by m, since we can always move the two integers that **are** divisible by m to the right side of the above equation and the remaining integer to the left side. Then there are some integers b and c such that $B = mb$ and $C = mc$.

Then $A = B + C = mb + mc = m(b + c)$ for some integers b and c. Thus A is divisible by m, too.

GENERATING DIVISIBILITY TESTS

We can use the Two out of Three theorem to generate formulas for divisibility tests.

The first set of divisibility tests we wish to consider is for numbers greater than 2 having a multiple that ends in 1 or 9. This is true of any number ending in 1 or 9, of course, but a little thought shows that it must also be true for numbers ending in 3 or 7 as well, because we can multiply these numbers by 3 or 7 and produce a multiple that ends in 1 or 9.

Let m be the putative factor of a given number. Then the assumption above, that m divides a number ending in 1 or 9, can be written algebraically:

(1) $m|(10n \pm 1)$, where "|" means "divides."

We want to devise a rule that tells us how to develop a divisibility test to ascertain whether a general number $(10a + b)$ is divisible by m.

Assume this is the case, i.e., that $m|(10a + b)$. Recall that we are in the case where m has a multiple ending in 1 or 9, that is, $m|(10n \pm 1)$. If we now let

$$B = 10a + b$$
$$C = (10n \pm 1)b$$

in the Two out of Three theorem, then $m|B$ and $m|C$. Then by the theorem, m must also divide

$$(2) \qquad A = B \mp C = (10a + b) \mp (10n \pm 1)b$$
$$A = 10a + b \mp 10nb - b = 10(a \mp nb)$$

So we have assumed that m divides a number $10n + 1$ or $10n - 1$ ending in 1 or 9 and that m also divides some number $B = 10a + b$. From this we have concluded that m will also divide the combination $a \mp nb$. Could this combination be our divisibility test?

The answer is yes. Use the Two out of Three theorem and equations (2) to show that if A and C are divisible by m, then B must be, also.

This leads to a divisibility test for factors of numbers of the form $10n + 1$ or $10n - 1$.

> Let a number m (sharing no factor with 10) be such that $m|(10n \pm 1)$ for some integer n. Take any number and perform the following transformation on it:
>
> Multiply the last digit by $\mp n$ and add to the rest of the number (i.e., the number formed by dropping the last digit).
>
> Then the result is divisible by m if and only if the original number is divisible by m.

(Question for the reader: Why do we need the restriction that m cannot share a factor with 10?)

This proves the following two claims from Goldenberg's paper. Goldenberg uses the notation u_n to mean the units digit of the number n we are testing, and r_n to mean the "rest" of the number, that is, the part that remains after the units digit is stripped off. Goldenberg also introduces the integer k (when it exists) by the condition that $ku_n = r_n$. In our notation, $a = r_n$ and $b = u_n$.

> **Claim:** $7|n$ if and only if $7|(r_n - 2u_n)$.
> **Claim:** $m|n$ if and only if $m|(r_n - ku_n)$.

Now, note that nowhere in our proof of the divisibility test did we assume that b was restricted to just one digit, although the divisibility tests where you use only one digit are easier to implement. So we have also proved the following (implied) claim from Goldenberg, where t means the last two digits of the number and h means the rest:

> **Claim:** This also works if you use h, t instead of r, u.

Let's return to our first example, divisibility by 41. Since 41 is a number of the form $10n + 1$, with $n = 4$, the divisibility test will be $a - 4b$. This explains why at each step of the test we multiply the last digit by 4 and subtract it from the rest of the number.

We now also understand how to construct **table 1**, which shows some of the divisibility tests for various values of p. Take $p = 13$, for example. Since $3 \cdot 13 = 39 = 4 \cdot 10 - 1$, one divisibility test for 13 is $a + 4b$. Since $7 \cdot 13 = 91 = 9 \cdot 10 + 1$, another test is $a - 9b$. The "4" and the "9" come from the multiple of 10 that a multiple of 13 is closest to. The "+" and "–" come from whether the multiple of 13 is one *less* than that multiple of 10 ("+") or one *more* ("–").

Again, note that p does not have to be prime for these divisibility tests to work. Nowhere in the Two out of Three theorem or the proof of the divisibility rules did we require that m was prime. As an example, since $3 \cdot 3 = 9 \cdot 1 = 1 \cdot 10 - 1$, the divisibility rule for 3, that is, "add the last digit to the rest of the number," also works for 9.

DIVISIBILITY TESTS FOR MULTIPLE NUMBERS

We constructed a divisibility rule for 13 above, using $3 \cdot 13 = 39 = 4 \cdot 10 - 1$ to get the test $a + 4b$. Rewriting this as $13 \cdot 3 = 39 = 4 \cdot 10 - 1$, we see that the test $a + 4b$ also is a test for divisibility by 3. This is an unexpected bonus.

For example, since $3 \cdot 7 = 21 = 2 \cdot 10 + 1$, the same test can be made to work for $p = 3$ and $p = 7$. This divisibility test for 3 or 7 is "multiply the last digit by 2 and subtract from the rest of the number."

Let's test 111111 for divisibility by 3 and by 7. Apply the test recursively, as follows:

$$111111 = 11111 \cdot 10 + 1$$
$$11111 - 1 \cdot 2 = 11109 = 1110 \cdot 10 + 9$$
$$1110 - 9 \cdot 2 = 1092 = 109 \cdot 10 + 2$$
$$109 - 2 \cdot 2 = 105 = 10 \cdot 10 + 5$$
$$10 - 5 \cdot 2 = 0$$

Since the result, 0, is divisible by both 3 and 7, the original number is, too. In fact, $111111 = 3 \cdot 7 \cdot 5291$.

The usefulness of this test is that it works independently for 3 and 7, so you are checking two numbers at once. [The editors pose this question to the reader: In this case, Olsen correctly sees the success of the test as showing that the number is divisible *both* by 7 *and* by 3. But clearly, not every

Table 2

Application to the Sieve of Eratosthenes

$a - 2b$	$b = 1$	$b = 3$	$b = 7$	$b = 9$
$a = 1$	–1	–5	–13	–17
$a = 2$	~~0~~	–4	~~–12~~	–16
$a = 3$	1	~~–3~~	–11	~~–15~~
$a = 4$	2	–2	–10	~~–14~~
$a = 5$	~~3~~	–1	~~–9~~	–13
$a = 6$	4	~~0~~	–8	~~–12~~
$a = 7$	5	1	~~–7~~	–11
$a = 8$	~~6~~	2	~~–6~~	–10
$a = 9$	~~7~~	~~3~~	–5	~~–9~~

number that passes the divisibility-by-7 test is also divisible by 3. What distinguishes when the process tests for divisibility by 3, by 7, or by both?]

You can easily construct other such tests. For example:

The test "$a - 9b$" checks both 7 and 13, as we saw above, since $7 \cdot 13 = 91$:

$$111111 = 11111 \cdot 10 + 1$$
$$11111 - 1 \cdot 9 = 11102 = 1110 \cdot 10 + 2$$
$$1110 - 2 \cdot 9 = 1092 = 109 \cdot 10 + 2$$
$$109 - 2 \cdot 9 = 91 = 9 \cdot 10 + 1$$
$$9 - 1 \cdot 9 = 0$$

Other examples are "$a + 4b$" for 3 and 13, and "$a - 5b$" for 3 and 17.

A useful classroom application of the (3, 7) test is the Sieve of Eratosthenes. With one test you can construct a list of prime numbers less than 121. If an odd number less than 121 does not end in a 5, then failing divisibility by both 3 and 7 means that it is a prime. [Why? Because any composite number less than 11^2 must have a prime less than 11 as a factor, and that leaves only 2, 3, 5, or 7 as possibilities. If 2 is a factor, the number is even; if 5 is a factor, the number must end in 5 or 0. Otherwise, the only possible factors are 3 or 7. . . unless the number itself is prime.—Eds.] For example, 31, 41, 61, and 71 are prime, but 51, 81, and 91 are not, as the test shows.

$$3 - 2 \cdot 1 = 1$$
$$4 - 2 \cdot 1 = 2$$
$$5 - 2 \cdot 1 = 3$$
$$6 - 2 \cdot 1 = 4$$
$$7 - 2 \cdot 1 = 5$$
$$8 - 2 \cdot 1 = 6$$
$$9 - 2 \cdot 1 = 7$$

A complete calculation is shown in **table 2**.

Suppose $n = 10a + b$. First we calculate $a - 2b$. Then we cross out entries that are divisible by 3 or 7. See **table 2**.

The remaining entries represent prime numbers: 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97 . . . Voilà!

One can also take these same divisibility tests, arrange them by multiples ending in 1 or 9, and then list their factors, as shown in **table 3**. This table shows, for example, that a divisibility test for 7 or 13 is to multiply the last digit by 9 and subtract it from the rest of the number.

Incidentally, note that the test for 99 means that if a two-digit number is divisible by 3, then the number formed by reversing the digits is, too. This, of course, conforms to the more common test for divisibility by 3: Sum all the digits and test that sum for divisibility by 3. Clearly, reversing the order of the digits would not affect that sum.

This finishes our survey of divisibility tests of numbers with multiples ending with 1 or 9. More questions arise, however: What about numbers ending with 3 or 7? Is there another set of tests that can be constructed explicitly for these numbers and their factors? The answer is yes, as we will see in the next section.

NUMBERS ENDING IN 3

Take a number like 13. One test for divisibility, above, is to multiply the units digit by 4 and add it to the rest of the number. It is tempting, though, to wonder if you could have a units test where you *divide* the units digit by 3 and subtract from the rest of the number. Let's see how it might work.

$$\begin{array}{r} 1|3 \\ \underline{-\,1} \\ 0 \end{array} \quad \text{Works . . .}$$

$$\begin{array}{r} 2|6 \\ \underline{-\,2} \\ 0 \end{array} \quad \text{Works . . .}$$

$$\begin{array}{r} 3|9 \\ \underline{-3} \\ 0 \end{array} \quad \text{We're on a roll!}$$

These work well, but what do we do with

$5|2?$ Ooops!

There are two ways to proceed here. First, to effect the division, we can borrow from the rest of the number. That is, 2 is not divisible by 3, but 12 is. So if we borrow one from the rest of the number,

$$5|2 \qquad \begin{array}{r} = 4|^{1}2 \\ \underline{-4} \\ 0 \end{array}$$

This seems to work. Regardless of the units digit, you have to borrow at most 2 from the rest of the number to find a multiple of 3.

The second method is to multiply the rest of the number by 3 and subtract the units, instead of dividing the units by 3 and subtracting from the rest of the number:

$$\begin{array}{r} 5|2 \\ 15|2 \\ \underline{-2} \\ 13 \end{array}$$

Either of these is satisfactory; however, the first method will converge faster because on average you are removing slightly more than one digit per operation.

GENERAL DIVISIBILITY TEST FOR FACTORS OF NUMBERS OF THE FORM 10*n* + 3 or 10*n* – 3

Now let's see if we can formalize this. If m divides $10n \pm 3$, the goal is to find a linear combination of $10n \pm 3$ and a general number $10a + b$ that will yield a divisibility test. In fact, if m is not a divisor of 10, then

$$A = (10a + b)$$

and $B = (10n \pm 3) \rightarrow$

$$C = 3A \mp B = 3(10a + b) \mp b(10n \pm 3)$$

or $C = 30a + 3b \mp 10nb - 3b = 10(3a \mp nb)$

$$C = 10 \bullet 3\left(a \mp \frac{n}{3}b\right)$$

Now, suppose that $m|B$ and $m|C$. Then by the Two out of Three theorem, $m|A$. This forms the basis of the following divisibility test.

Let a number m, relatively prime to 10 and not divisible by 3, be a divisor of $10n \pm 3$. Take any number $10a + b$ and perform the following transformation on it:

Borrow 0, 1, or 2 from a so that b, $b + 10$, or $b + 20$ respectively is evenly divisible by 3.

Multiply the modified value of b by

$$k = \mp\frac{n}{3}$$

and add to the rest of the number (i.e., a, the number formed by taking off the last digit).

Then the result is divisible by m if and only if the original number is divisible by m.

Table 3

Divisibility Tests for Numbers Ending in 1 or 9

n	Test for	k
9	3, 9	+ 1
11	11	– 1
19	19	+ 2
21	3, 7, 21	– 2
29	29	+ 3
31	31	– 3
39	3, 13, 39	+ 4
41	41	– 4
49	7, 49	+ 5
51	3, 17, 51	– 5
59	59	+ 6
61	61	– 6
69	3, 23, 69	+ 7
71	71	– 7
79	79	+ 8
81	3, 9, 27, 81	– 8
89	89	+ 9
91	7, 13, 91	– 9
99	3, 9, 11, 99	+ 10

Examples of these tests are shown in **table 1** and summarized in **table 4**.

As an example, let's prove that the repeat length of 1/13 is 6. As demonstrated in the first part of the article, this comes down to showing that 111111 is divisible by 13. We use the test given in **table 4**, which is (–1/3).

$11111\|1$	Starting number
$11109\|^{2}1$	Borrow 2
$\underline{-7}$	1/3 of 21 is 7; subtract it
$1110\|2$	Second number
$1109\|^{1}2$	Borrow 1
$\underline{-4}$	1/3 of 12 is 4; subtract it
$110\|5$	Third number

Table 4

Testing Numbers Ending in 3 or 7

p	*k*
7	+ 1/3
13	– 1/3
17	+ 2/3
23	– 2/3
37	+ 4/3
43	– 4/3
47	+ 5/3
53	– 5/3
67	+ 7/3
73	– 7/3
83	– 8/3
97	+ 10/3

$109 \mid {}^{1}5$	Borrow 1
– 5	1/3 of 15 is 5; subtract it
$10 \mid 4$	Fourth number
$8 \mid {}^{2}4$	Borrow 2
– 8	1/3 of 24 is 8; subtract it
0	0, 104, 1105, 11102, and 111111 are all divisible by 13

We could have used any test, for example, (+ 4) from **table 1**:

$11111 \mid 1$	Starting number
+ 4	Multiply by 4, add
$1111 \mid 5$	
+ 20	5 times 4 is 20
$113 \mid 1$	
+ 4	1 times 4 is 4
$11 \mid 7$	
+ 28	7 times 4 is 28
39	Yes, 111111 is divisible by 13.

APPLICATIONS

This completes my survey of divisibility tests. I want to close by briefly mentioning some applications.

The first application, mentioned above, is in middle school classrooms, where students learn about the Sieve of Eratosthenes. The single divisibility test for 3 and 7 can be used quickly to find all the primes less than 121.

A second application is factoring of numbers and polynomials. Factoring polynomials starts by factoring the constant term. For example, to factor $x^2 - 6x - 1591$, you would start by factoring 1591. My algebra students have found divisibility tests useful in quickly ruling out certain combinations. Here you are looking for two numbers close to $40 = \sqrt{1600}$ that divide 1591. Try 41. **Table 1** says to apply "– 4":

$159 \mid 1$	
– 4	
$15 \mid 5$	
– 20	
– 5	Not divisible by 41

Try 43:

$159 \mid 1$	Use the (–4/3) test from **table 4**
$157 \mid {}^{2}1$	Borrow 2
– 28	4/3 times 21 is 28
$12 \mid 9$	
– 12	4/3 times 9 is 12
0	1591 is divisible by 43

The other factor is 37, which you can find by division or by

$159 \mid 1$	Use the (+4/3) test from **table 4**
$157 \mid {}^{2}1$	Borrow 2
+ 28	4/3 times 21 is 28
$18 \mid 5$	
$17 \mid {}^{1}5$	Borrow 1
+ 20	
37	1591 is divisible by 37

I have found that once students master these tests, they can apply them faster than using a calculator to divide the number they are trying to factor.

What is interesting about these methods, from a mathematical perspective, is that they transform a very hard problem (factoring) to a much easier one by reducing the size of the number that needs factoring. Factoring algorithms form the computational basis of much of the data encryption standards used by the financial community to safeguard commercial transactions. Is it too much to hope that these methods could be applied to factoring very large numbers (thousands of digits) to improve the performance of these algorithms?

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Finally, special thanks to Al Cuoco for his interest and encouragement, for the many conversations and e-mails we have exchanged, and for his steadfast belief that some day an article would come of all this.

Editors' notes: Allen Olsen picks up the theme from Goldenberg's article and shows how to develop divisibility tests for any integer that has a multiple of the form $10n \pm 1$ or $10n \pm 3$. It turns out that every integer that is not divisible by 2 or 5 satisfies all four of these conditions. The reason comes from number theory: If m is an integer relatively prime to 10, then the greatest common divisor for m and 10 is 1. It follows from a famous theorem that appears in Euclid that one can find an integer n so that $10n$ leaves a remainder of 1 when divided by m (n is called the *inverse* of 10 modulo m). From here, you can find multiples of 10 that leave remainders of –1, 3, and –3 when divided by m. So there are several tests for all such integers m.

We have seen other examples of applications of Euclid's result and number theory in "Delving Deeper," such as "Reducing the Sum of Two Fractions," by Harris S. Shultz and Ray C. Shiflett, *Mathematics Teacher* 98 (March 2005): 486–90. We would welcome other articles that explore this beautiful part of mathematics. For example, readers may have used the "post office" problem in their classes ("If the post office has only 5- and 8-cent stamps today, what denominations can they make?"). What mathematics can you mine from this problem?

As Olsen mentions, one topic closely related to this circle of ideas is the analysis of repeat lengths in the decimal expansions for rational numbers. This topic fascinates people of all ages (see the "Reader Reflection" by Walt Levisee in the March 1997 issue for an account of work done by a nine-year-old). A great deal of insight into this topic comes from a thoughtful analysis of long division. For example, here is the calculation for the decimal expansion for 1/7:

One topic closely related to this circle of ideas is the analysis of repeat lengths in the decimal expansions for rational numbers

$$
\begin{array}{r}
.142857 \\
7\,\overline{)1.00000} \\
\underline{7} \\
30 \\
\underline{28} \\
20 \\
\underline{14} \\
60 \\
\underline{56} \\
40 \\
\underline{35} \\
50 \\
\underline{49} \\
1
\end{array}
$$

From here on it repeats, because the next step is to divide 7 into 1 again; so the period is 6. But another way to look at this calculation (if you forget the decimal point) is that 10^6 leaves a remainder of 1 when divided by 7. This leads to more questions: For what integers m is there a power of 10 that leaves a remainder of 1 when divided by m? And how can this help determine the length of the period for $1/m$? Again, we would be happy to see articles that address these ideas.

As to the speculation that the approach presented here might find application in factoring large numbers, we are skeptical. For one thing, removing the units digit from a number is not easy in the multiprecision environment of most computers.

The Lawrence High School Study Group had another visitor on the day that Paul Goldenberg presented his results: Kurt Kreith from the University of California—Davis. Kreith raised an interesting point: In the common divisibility test for 3, one computes the sum of the digits of a number and tests it for divisibility by 3. More is true: The number and the sum of its digits leave the same remainder when each is divided by 3. This is not true for, say, Olsen's divisibility test for 7. All Olsen's test guarantees is that if one remainder on division by 7 is 0, so is the next one in the process. Can one say *anything* about the remainders in this situation? ∞

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