

The Chebyshev Polynomials: Patterns and Derivation

Pafnuty Lvovich Chebyshev, a Russian mathematician, is famous for his work in the area of number theory and for his work on a sequence of polynomials that now bears his name. These *Chebyshev polynomials* have applications in the fields of polynomial approximation, numerical analysis, graph theory, Fourier series, and many other areas. They can be derived directly from the multiple-angle formulas for sine and cosine. They are relevant in high school and in the broader mathematical community. For this reason, the Chebyshev polynomials were chosen as one of the topics for study at the 2003 High School Teachers Program at the Park City Mathematics Institute (PCMI). The following is a derivation of the Chebyshev polynomials and a mathematical exploration of the patterns that they produce.

MULTIPLE-ANGLE FORMULAS

Many of us have committed multiple-angle formulas for the sine and cosine to memory or know some ways to derive them. The most familiar are the double-angle formulas for sine ($\sin 2\theta = 2 \sin \theta \cos \theta$) and cosine ($\cos 2\theta = 2 \cos^2 \theta - 1$). One way to derive these formulas is to use the multiplication rule for complex numbers. A method for deriving this rule without trigonometry is in Kerins and the High School Teachers Program Group of the Park City Mathematics Institute (2003). The basic fact that we need is

$$(1) [\cos \alpha + i \sin \alpha] \cdot [\cos \beta + i \sin \beta] = \cos (\alpha + \beta) + i \sin (\alpha + \beta).$$

A more compact notation for equation (1) is

$$(2) \quad \text{cis } \alpha \text{ cis } \beta = \text{cis } (\alpha + \beta),$$

where $\text{cis } x = \cos x + i \sin x$. We can use formula (2) to derive multiple-angle formulas. The detailed derivations of the double- and triple-angle formulas follow.

Double-angle formula

Substituting $\alpha = \beta = \theta$ in formula (2) gives

$$\text{cis } \theta \cdot \text{cis } \theta = \text{cis } 2\theta.$$

Doing some algebra on the left side of the equation gives

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Chebyshev polynomials have applications in many areas

$$(\cos^2 \theta - \sin^2 \theta) + 2i \sin \theta \cos \theta = \cos 2\theta + i \sin 2\theta.$$

Substituting $\sin^2 \theta = 1 - \cos^2 \theta$ (from the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$) gives

$$2 \cos^2 \theta - 1 + i(2 \sin \theta \cos \theta) = \cos 2\theta + i \sin 2\theta.$$

Separating the real part from the imaginary part gives the double-angle formulas,

$$\cos 2\theta = 2 \cos^2 \theta - 1$$

and

$$\sin 2\theta = 2 \sin \theta \cos \theta.$$

Triple-angle formula

Substituting $\alpha = 2\theta$ and $\beta = \theta$ into equation (2) gives

$$\text{cis } 2\theta \cdot \text{cis } \theta = \text{cis } 3\theta.$$

Substituting the double-angle formulas for $\text{cis } 2\theta$ and doing some algebra give

$$\begin{aligned} 2 \cos^3 \theta - \cos \theta - 2 \sin^2 \theta \cos \theta \\ + 2i \sin \theta \cos^2 \theta + 2 \sin \theta i \cos^2 \theta - i \sin \theta \\ = \cos 3\theta + i \sin 3\theta. \end{aligned}$$

Just as in the derivation of the double-angle formula, we can substitute $\sin^2 \theta = 1 - \cos^2 \theta$ to get

$$\begin{aligned} 4 \cos^3 \theta - 3 \cos \theta + (4 \sin \theta \cos^2 \theta - \sin \theta)i \\ = \cos 3\theta + i \sin 3\theta. \end{aligned}$$

We now have the triple-angle formulas for cosine and sine:

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$$

and

$$\sin 3\theta = 4 \sin \theta \cos^2 \theta - \sin \theta$$

Multiple-angle formulas

By following steps similar to the ones for the double-angle formula and the triple-angle formula, we can obtain all the multiple-angle formulas for sine and cosine. We next focus on the polynomials formed from the multiple-angle formulas for cosine. The first few multiple-angle formulas for the cosine are

$$\cos 0\theta = 1$$

$$\cos 1\theta = \cos \theta$$

$$\cos 2\theta = 2 \cos^2 \theta - 1$$

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$$

$$\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$$

$$\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$$

$$\cos 6\theta = 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1.$$

If we look at how we might proceed from the formula for $\cos(k-1)\theta$ to that for $\cos k\theta$, it seems that we need to multiply the formula for $\cos(k-1)\theta$ by $2 \cos \theta$ and subtract a little bit. In fact, the little bit that we need to subtract is $\cos(k-2)\theta$. For example,

$$\begin{aligned} \cos 2\theta &= 2 \cos^2 \theta - 1 \\ &= 2 \cos \theta (\cos \theta) - 1 \\ &= 2 \cos \theta (\cos 1\theta) - \cos(0\theta). \end{aligned}$$

Another example is

$$\begin{aligned} \cos 4\theta &= 8 \cos^4 \theta - 8 \cos^2 \theta + 1 \\ &= 2 \cos \theta (4 \cos^3 \theta - 3 \cos \theta) - (2 \cos^2 \theta - 1) \\ &= 2 \cos \theta (\cos 3\theta) - \cos 2\theta. \end{aligned}$$

Perhaps we now have a recursive method by which we can make multiple-angle formulas:

Let

$$t_k(x) = \begin{cases} 1 & \text{if } k = 0 \\ x & \text{if } k = 1 \\ 2 \cdot x \cdot t_{k-1}(x) - t_{k-2}(x) & \text{if } k > 1 \end{cases}$$

When $x = \cos \theta$, we claim that $t_k(\cos \theta) = \cos k\theta$. The proof, using DeMoivre's theorem, is given in the next section. The following is a list of the first few $t_k(x)$:

$$t_0(x) = 1$$

$$t_1(x) = x$$

$$t_2(x) = 2x^2 - 1$$

$$t_3(x) = 4x^3 - 3x$$

$$t_4(x) = 8x^4 - 8x^2 + 1$$

$$t_5(x) = 16x^5 - 20x^3 + 5x$$

$$t_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

$$t_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$

These polynomials are formally known as the *Chebyshev polynomials of the first kind*; in this article, we call them the *Chebyshev polynomials*.

SKETCH OF A PROOF

DeMoivre's theorem implies that $(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta$. This result offers us a tool that we

can use to prove that $t_k(\cos\theta) = \cos k\theta$. We start with the following algebraic identity:

$$(3) \alpha^k + \beta^k = (\alpha + \beta)(\alpha^{k-1} + \beta^{k-1}) - \alpha\beta(\alpha^{k-2} + \beta^{k-2})$$

Next, we let $\alpha = \cos \theta + i \sin \theta$ and let $\beta = \cos \theta - i \sin \theta$, the complex conjugate of α . Then $\alpha\beta = 1$ and $\alpha + \beta = 2 \cos \theta$. Substituting these values into equation (3) gives

$$(4) \alpha^k + \beta^k = 2 \cos \theta (\alpha^{k-1} + \beta^{k-1}) - (\alpha^{k-2} + \beta^{k-2}).$$

We use DeMoivre's theorem, the facts that the sine is an odd function and that the cosine is an even function, and substitution to get

$$\begin{aligned} (\cos k\theta + i \sin k\theta) + (\cos k\theta - i \sin k\theta) \\ = 2 \cos \theta ((\cos(k-1)\theta + i \sin(k-1)\theta) \\ + (\cos(k-1)\theta - i \sin(k-1)\theta)) \\ - (\cos(k-2)\theta + i \sin(k-2)\theta) \\ - (\cos(k-2)\theta - i \sin(k-2)\theta). \end{aligned}$$

By simplifying and dividing both sides of the equation by 2, we get $\cos k\theta = 2 \cos \theta (\cos(k-1)\theta) - \cos(k-2)\theta$.

So $\cos k\theta$ satisfies the same recursion as $t_k(x)$, and it follows by induction on k that $t_k(\cos \theta) = \cos k\theta$.

One way to generate the Chebyshev polynomials on a computer algebra system (CAS) is to use the Pythagorean identity. If $x = \cos \theta$, then $\sin \theta = \sqrt{1 - x^2}$.

Substituting $\sqrt{1 - x^2}$ for $\sin \theta$ and x for $\cos \theta$ into $\text{cis } \theta$ gives

$$(5) \quad x + i \sqrt{1 - x^2}.$$

When identity (3) is raised to the k th power, the real part yields the k th Chebyshev polynomial. Expression (5) can be quickly expanded with a CAS.

PATTERNS IN THE GRAPHS

When graphed, the Chebyshev polynomials produce some interesting patterns. **Figure 1** shows the first four Chebyshev polynomials, and **figure 2** shows the next four.

The following patterns can be discerned by analyzing these graphs. Even-numbered Chebyshev polynomials yield even functions whose graphs have reflective symmetry across the y -axis. Odd-numbered Chebyshev polynomials yield odd functions whose graphs have 180-degree rotational symmetry around the origin. The proofs of these symmetries follow by induction from the way that the polynomials are generated. All the zeros for Chebyshev polynomials are between -1 and 1 . In fact, because $t_k(\cos \theta) = \cos k\theta$, the zeros of the k th Chebyshev polynomial are of the form $\cos \theta$, where $\cos k\theta = 0$. Since the cosine is 0 at odd multiples of $\pi/2$, the zeros of $t_k(x)$ are of the form

$$\cos \left(\frac{(2q-1) \cdot \frac{\pi}{2}}{k} \right),$$

where $1 \leq q \leq 2k - 1$. A detailed derivation of this "perfect example of where form and function come together for polynomials" can be found in *Mathematical Connections* (Cuoco forthcoming).

PATTERNS IN A TABLE

By using the list of Chebyshev polynomials given previously, we can look for patterns in the coefficients. We define the integers $t(k, j)$ with the sequence shown in **figure 3**.

In general, the $t(k, j)$ are defined by

$$t_k(x) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} t(k, j) x^{k-2j},$$

where

$$\left\lfloor \frac{k}{2} \right\rfloor$$

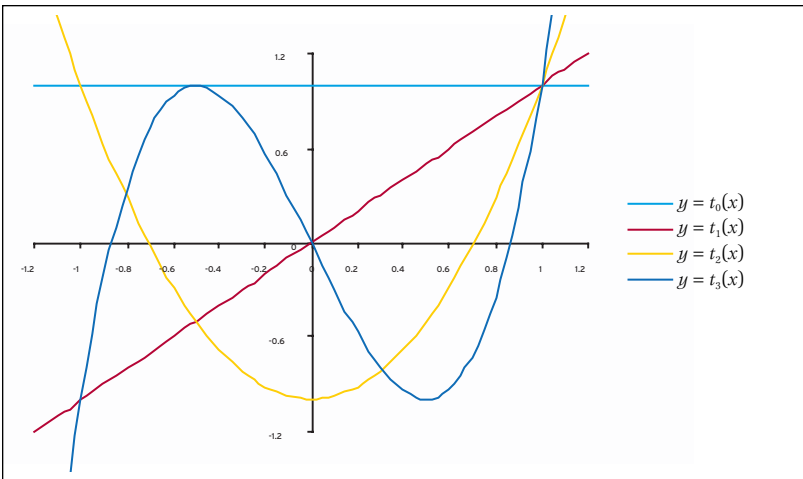


Fig. 1 The graphs of the first four Chebyshev polynomials

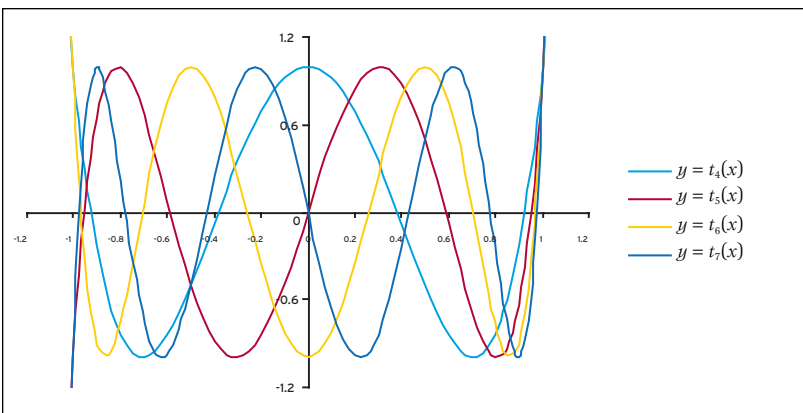


Fig. 2 The graphs of the second four Chebyshev polynomials

is the greatest integer less than or equal to $k/2$.

Table 1 shows the first few $t(k, j)$.

Many patterns exist throughout **table 1**, both within and between the columns and rows. One pattern is that any coefficient seems to be the sum of all the numbers above it minus the sum of all the numbers diagonally up and to the left of it. For example, to get $t(8, 2)$, the third coefficient of the eighth Chebyshev polynomial, we add the numbers in red; and from that sum, we subtract the sum of the numbers in dark blue (not including column k , which is just there to keep track of the row). Since each cell in the table seems to obey this rule, we can see that some “serious” recursion is going on. Many other patterns can be found in the table. We focus on finding functions that fit each column.

A GENERAL (SUMMATION) FORMULA

If we look at our Chebyshev coefficients, we see that the sign alternates across any row, so we can factor out $(-1)^j$ from each coefficient. If we look some more, we see that odd powers of 2 can be factored from each coefficient. If we look at the $k = 6$ row in **table 1**, we see the values shown in **figure 4**.

We can factor out a 2^5 from $t(6, 0)$, a 2^3 from $t(6, 1)$, and a 2^1 from $t(6, 2)$.

To complete the pattern of odd exponents, we factor out a 2^{-1} from $t(6, 3)$, as shown in **figure 5**.

In general, $2^{k-(2j+1)}$ can be factored from each coefficient. The reason can be seen in the recursion, where each term ($k \geq 2$) is multiplied by 2. So we can write

$$t_k(x) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j 2^{k-(2j+1)} a(k, j) x^{k-2j}.$$

Table 2 lists the general form of the factors of -1 and the powers of 2 for the first five coefficients ($j = 0$ to $j = 4$) and for the j th coefficient of the k th Chebyshev polynomial.

Table 3 shows what is left after factoring out $(-1)^j$ and $2^{k-(2j+1)}$ from $t(k, j)$.

The relatively simple polynomials (in terms of k), shown in **table 4**, fit the nonzero elements of each column in **table 3** with the exception of $t(0, 0)$. We discuss the exception a little later.

The polynomials in **table 4** were obtained by using Newton’s difference formula, a topic discussed in Cuoco (2003). These polynomials fit the nonzero data in each column. The tricky part here is writing a general term for the polynomials. With some effort, much experimenting, and some help from the CAS, we see that

$$a(k, j) = \frac{\binom{2j}{j} \binom{k}{2j}}{\binom{k-1}{j}}$$

$t_0(x) = 1$	$= t(0, 0)x^0$
$t_1(x) = x$	$= t(1, 0)x^1$
$t_2(x) = 2x^2 - 1$	$= t(2, 0)x^2 + t(2, 1)x$
$t_3(x) = 4x^3 - 3x$	$= t(3, 0)x^3 + t(3, 1)x$
$t_4(x) = 8x^4 - 8x^2 + 1$	$= t(4, 0)x^4 + t(4, 1)x^2 + t(4, 2)x^0$
$t_5(x) = 16x^5 - 20x^3 + 5x$	$= t(5, 0)x^5 + t(5, 1)x^3 + t(5, 2)x$
$t_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$	$= t(6, 0)x^6 + t(6, 1)x^4 + t(6, 2)x^2 + t(6, 3)x^0$
$t_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$	$= t(7, 0)x^7 + t(7, 1)x^5 + t(7, 2)x^3 + t(7, 3)x$

Fig. 3 Looking for patterns in the coefficients of the Chebyshev polynomials

TABLE 1

The First Five Coefficients for the First Ten Chebyshev Polynomials

k	$t(k, 0)$	$t(k, 1)$	$t(k, 2)$	$t(k, 3)$	$t(k, 4)$	$t(k, 5)$
0	1	0	0	0	0	0
1	1	0	0	0	0	0
2	2	-1	0	0	0	0
3	4	-3	0	0	0	0
4	8	-8	1	0	0	0
5	16	-20	5	0	0	0
6	32	-48	18	-1	0	0
7	64	-112	56	-7	0	0
8	128	-256	160	-32	1	0
9	256	-576	432	-120	9	0
10	512	-1280	1120	-400	50	-1

k	$t(k, 0)$	$t(k, 1)$	$t(k, 2)$	$t(k, 3)$	$t(k, 4)$	$t(k, 5)$
·	·	·	·	·	·	·
·	·	·	·	·	·	·
·	·	·	·	·	·	·
6	32	-48	18	-1		

Fig. 4 The $k = 6$ row of table 1

k	$t(k, 0)$	$t(k, 1)$	$t(k, 2)$	$t(k, 3)$	$t(k, 4)$	$t(k, 5)$
·	·	·	·	·	·	·
·	·	·	·	·	·	·
·	·	·	·	·	·	·
6	$2^5 \cdot 1$	$2^3 \cdot (-16)$	$2^1 \cdot 9$	$2^{-1} \cdot (-2)$		

Fig. 5 Factoring odd powers of 2 from the $k = 6$ row of table 1

$$\text{when } 0 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor.$$

For example, when $j = 3$, we get

$$a(k, 3) = \frac{\binom{6}{3} \binom{k}{6}}{\binom{k-1}{3}} = \frac{k(k-4)(k-5)}{3!}.$$

Since we know how to obtain each term for a given Chebyshev polynomial, we can write them in summation notation. The formula is (again, for the k th Chebyshev polynomial, where $k \neq 0$)

$$\sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \cdot \frac{\binom{2j}{j} \binom{k}{2j}}{\binom{k-1}{j}} \cdot 2^{k-(2j+1)} \cdot x^{k-2j}.$$

This formula makes calculating the k th Chebyshev polynomial relatively easy with a CAS. When $k = 0$, this formula gives $t_0(x) = 0$, instead of $t_0(x) = 1$. To resolve this discrepancy, we need to find a way of adding a term to the summation formula that is 1 when $k = 0$ but that is 0 at all other times:

$$\binom{0}{k}$$

meets both these criteria. So, to make the summa-

tion formula work when $k = 0$, we add

$$\binom{0}{k}$$

to the result of the summation.

To find a closed formula for Chebyshev polynomials, we can use the method of generating functions.

DELVING DEEPER

First, we can make the Chebyshev polynomials coefficients of a power series in z :

$$F(z) = 1 + xz + (2x - 1)z^2 + (4x^3 - 3x)z^3 + (8x^4 - 8x^2 + 1)z^4 + \dots$$

$F(z)$ is the generating function for the Chebyshev polynomials; and the coefficient of z^k equals $t_k(x)$, the k th Chebyshev polynomial. Multiplying $F(z)$ by $2xz$ gives

$$2xzF(z) = 2xz + (2x^2)z^2 + (4x^3 - 2x)z^3 + (8x^4 - 6x^2)z^4 + \dots$$

Subtracting $2xzF(z)$ from $F(z)$ gives

$$(1 - 2xz)F(z) = 1 - xz - (1)z^2 - (x)z^3 - (2x^2 - 1)z^4 + \dots$$

We can see that

$$-(1)z^2 - (x)z^3 - (2x^2 - 1)z^4 + \dots$$

is $-z^2F(z)$. Substituting gives

$$(1 - 2xz)F(z) = 1 - xz - z^2F(z).$$

By solving for $F(z)$, we get

$$F(z) = \frac{1 - xz}{1 - 2xz + z^2}.$$

We know that the roots of the denominator (by the quadratic formula) are

$$\alpha = x + \sqrt{x^2 - 1}$$

and

$$\beta = x - \sqrt{x^2 - 1}.$$

TABLE 2 General Form of the Factors of -1 and the Powers of 2 for the First Five Coefficients ($j = 0$ to $j = 4$)					
$t(k, 0)$	$t(k, 1)$	$t(k, 2)$	$t(k, 3)$	$t(k, 4)$	$t(k, j)$
$(-1)^0 2^{k-1}$	$(-1)^1 2^{k-3}$	$(-1)^2 2^{k-5}$	$(-1)^3 2^{k-7}$	$(-1)^4 2^{k-9}$	$(-1)^j 2^{k-(2j+1)}$

TABLE 3 Remaining Values after Factoring $(-1)^j$ and $2^{k-(2j+1)}$ from $t(k, j)$						
k	$a(k, 0)$	$a(k, 1)$	$a(k, 2)$	$a(k, 3)$	$a(k, 4)$	$a(k, 5)$
0	2	0	0	0	0	0
1	1	0	0	0	0	0
2	1	2	0	0	0	0
3	1	3	0	0	0	0
4	1	4	2	0	0	0
5	1	5	5	0	0	0
6	1	6	9	2	0	0
7	1	7	14	7	0	0
8	1	8	20	16	2	0
9	1	9	27	30	9	0
10	1	10	35	50	25	2

TABLE 4 Relatively Simple Polynomials (in Terms of k) That Fit the Elements of Each Column in Table 3					
$a(k, 0)$	$a(k, 1)$	$a(k, 2)$	$a(k, 3)$	$a(k, 4)$	$a(k, j)$
$\frac{1}{0!}$	$\frac{k}{1!}$	$\frac{k(k-3)}{2!}$	$\frac{k(k-4)(k-5)}{3!}$	$\frac{k(k-5)(k-6)(k-7)}{4!}$	$\frac{k(k-(j+1)) \dots (k-(2j-1))}{j!}$

If we do the partial fraction decomposition, we get

$$\frac{1 - xz}{1 - 2xz + z^2} = A \left(\frac{1}{1 - \frac{1}{\alpha}z} \right) + B \left(\frac{1}{1 - \frac{1}{\beta}z} \right),$$

where A and B are constants. The parts in the parentheses are the sums of geometric series. For example,

$$\frac{1}{1 - \frac{1}{\alpha}z} = 1 + \frac{1}{\alpha}z + \frac{1}{\alpha^2}z^2 + \frac{1}{\alpha^3}z^3 + \dots$$

So the formula for the coefficient of the k th term (the k th Chebyshev polynomial) in $F(z)$ is

$$A \left(\frac{1}{\alpha} \right)^k + B \left(\frac{1}{\beta} \right)^k.$$

Since we know that the 0th Chebyshev polynomial ($k = 0$) is 1 and that the first Chebyshev polynomial ($k = 1$) is x , we can solve the system

$$\begin{cases} A \left(\frac{1}{\alpha} \right)^0 + B \left(\frac{1}{\beta} \right)^0 = 1 \\ A \left(\frac{1}{\alpha} \right)^1 + B \left(\frac{1}{\beta} \right)^1 = x \end{cases}$$

for A and B . After doing some algebra, we obtain $A = 1/2$ and $B = 1/2$. Thus, the closed formula for the k th Chebyshev polynomial is

$$\frac{1}{2} \left(\frac{1}{x + \sqrt{x^2 - 1}} \right)^k + \frac{1}{2} \left(\frac{1}{x - \sqrt{x^2 - 1}} \right)^k.$$

CONCLUSION

The preceding derivation and exploration of patterns in the Chebyshev polynomials has covered a range of topics that are relevant to the high school classroom and beyond. The use of a CAS makes many of these topics more accessible, even for high school students. The Chebyshev polynomials contain a wealth of patterns and applications that have not been presented here. Some other topics left to the reader include relating the Chebyshev polynomials to the Fibonacci numbers; finding a set of matrices whose determinants give the Chebyshev polynomials, similar to what was done in Askey (2004); the Chebyshev polynomials of the second kind; and finding closed formulas for recursive functions.

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